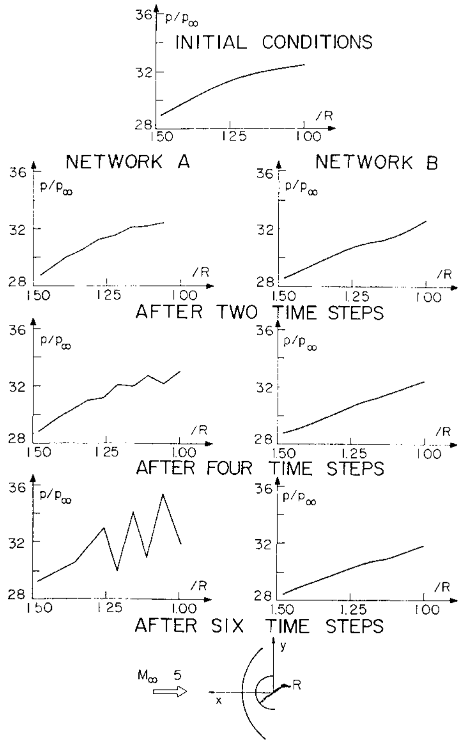

Fig 3 The pressure near the axis of symmetry for a blunt-body flow

of characteristics using network A was first used to calculate the flow with the body surface held constant in time to check that the initial solution would hold steady as the flow was calculated at later times It was found that the pressure along the axis of symmetry was the most sensitive indicator of the onset of instability Figure 3 shows plots of the pressure ratio vs the coordinate parallel to the freestream velocity for points closest to the axis after various time steps in the calculation Note that these points are not located exactly on the axis and do not have the same space coordinates at each step so that the magnitudes of the pressure ratio should not be exactly the same on each plot; however, they do indicate very graphically the onset of instability Network B then was used to repeat the calculation with the results shown on the right side of Fig 3 The instability with network A had grown so large by the eighth step that calculation could not be continued, whereas no instability has been detected while using the modified network B

Conclusions

Previously, it has been tacitly assumed in three-dimensional characteristic calculations that integration on the characteristic surface, or at least along bicharacteristic lines, assures the convergence and stability of the process This is not the case Care must be taken in choosing a net configuration that is stable and convergent The Courant Friedrichs-Lewy condition can be employed as a test for stability and convergence of a finite difference network that is simplicial, and it can be argued physically that this provides a necessary condition for the stability of networks that are not simplicial

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Elastic Stability of Conical Shells under Axisymmetric Pressure Band Loadings

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In a recent publication,[1] the buckling characteristics of circular cones and cylinders subjected to axisymmetric loadings were investigated Typical results were illustrated for various loadings and compared with previous analytical and experimental studies

This paper is concerned with applying the theoretical equations developed in Ref 1 to assess the stability response of simply supported cones and cylinders under circumferential band loadings One important purpose of the present investigation is to illustrate a simplified method for estimating the critical pressures for circumferential band loadings Another objective is to demonstrate that the predictions based on such a method are satisfactorily correlated with those rigorously obtained by Almroth and Brush[2] for the limiting case of a circumferential band loading that is uniformly distributed and equally spaced between cylinder supports

The value of such a correlation is that it lends credence to the present method of approach and, further, indicates the usefulness of the Donnell-type theory used in Ref 1 for estimating the stability response for arbitrary, axisymmetric loading conditions The analysis now is extended for the simply supported cylinder under a uniformly distributed and equally spaced, circumferential band loading

The equation governing the elastic buckling of circular cylinders subjected to axisymmetric loadings is given by [e g, Ref 1, Eq (28)]

$$\bar{P} = [\bar{\Phi}^2 + 1/\bar{\Phi}^2]/(\bar{\Phi} + \bar{S}/2) \tag{1}$$

where

$$\begin{aligned} \bar{P} &= \bar{p}_c K_2 12(1 - \nu^2)/\lambda^2 \bar{H}^{1/4} E h^3 \\ \bar{\Phi} &= (1 + n^2/\lambda^2 a^2)/\bar{H}^{1/4} \\ \bar{S}/2 &= [(K_1/a^2 K_2) - 1]/\bar{H}^{1/4} \\ \bar{H} &= 12\,(1 - \nu^2)/\lambda^4 a^2 h^2 \\ \lambda &= m\pi/L \end{aligned} \tag{2}$$

and

$$K_1 = -\frac{2a^2}{L}\int_0^L \left(\frac{\bar{N}_\xi}{p}\right)\cos^2\lambda\xi d\xi$$

$$K_2 = -\frac{2}{L}\int_0^L \left(\frac{\bar{N}_\theta}{p}\right)\sin^2\lambda\xi d\xi \tag{3}$$

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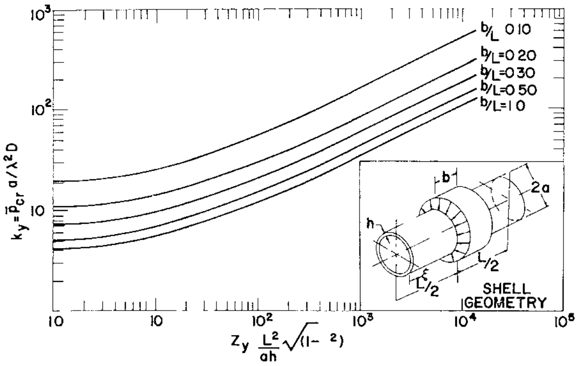

Fig 1 Graphical results for critical band pressure

The minimum value of the critical load can be determined for the present application of lateral pressure loadings by equating the wave number $m = 1$ and by formally minimizing $\bar{P}$ with respect to the parameter $\bar{\phi}$ The results are†

$$\bar{P} = [\alpha^2 + 1/\alpha^2]/(\alpha + \bar{S}/2) \tag{4}$$

in which

$$\alpha = \left[\frac{-\beta + (\beta^2 + 12)^{1/2}}{2}\right]^{1/2} \qquad \beta = \bar{P}\bar{S}/2 \tag{5}$$

The predictions given by Eq (4) are essentially the same as those developed in Eq (30) of Ref 1 except that the present results are exact, and, as such, the constraining conditions placed on Eq (30) have been removed

From an inspection of Eq (4), it is seen that the extension of results to the special case of circumferential band loadings only requires an evaluation of the parameters K_1 and K_2 defined by Eqs (3) If the remarks are confined to a uniform-pressure band loading with no axial forces acting, the parameter K_1 vanishes identically

Introducing the following Heaviside step function to describe the pressure loading (Fig 1), we have

$$p(\xi) = -p\left\{H\left[\xi - \left(\frac{L-b}{2}\right)\right] - H\left[\xi - \left(\frac{L+b}{2}\right)\right]\right\} \tag{6}$$

or

$$\bar{N}_\theta = -pa\left\{H\left[\xi - \left(\frac{L-b}{2}\right)\right] - H\left[\xi - \left(\frac{L+b}{2}\right)\right]\right\} \tag{7}$$

n which

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \tag{8}$$

Substituting Eq (7) into (3) and integrating, K_2 becomes

$$K_2 = (a/\pi)[\pi b/L + \sin(\pi b/L)] \tag{9}$$

Expanding $\bar{P}$ and $\bar{S}$ in terms of parameters used previously by Almroth,[2] we have

$$\bar{P} = \frac{[12(1-\nu^2)]^{3/4}}{\pi^2}\left(\frac{\bar{p}_{cr}a}{Eh}\right)\left(\frac{a}{h}\right)^{3/2}\left(\frac{L}{a}\right)\left[\frac{\pi b}{L} + \sin\left(\frac{\pi b}{L}\right)\right] \tag{10}$$

$$\frac{\bar{S}}{2} = -\frac{\pi}{[12(1-\nu^2)]^{1/4}}\left(\frac{a}{L}\right)\left(\frac{h}{a}\right)^{1/2} \tag{11}$$

If attention is now drawn to those ranges for which $10^2 < (L/a)^2\,(a/h) < 10^4$,‡ some interesting simple results are obtained That is, the parameter $\bar{S}$ approaches zero in Eq (11), in which case Eq (4) becomes, after an appropriate substitution for $\bar{P}$,

$$\frac{p_{cr}a}{Eh} = \frac{4\pi^2\,3^{1/4}}{3[12(1-\nu^2)]^{3/4}}\,\frac{(h/a)^{3/2}\,(a/L)}{[\pi b/L + \sin(\pi b/L)]} \tag{12}$$

From an inspection of Eq (12), it is seen that, for fixed (L/a) ratios and for (b/L) small in comparison to unity, the critical pressure parameter, $(\bar{p}_c\,a/Eh)$, varies inversely as $(a/h)^{3/2}$ and (b/L) This accounts for the fact that, in logarithmic graphs presented by Almroth,[2] the arithmetic slopes of the $(\bar{p}\;a/Eh)$ vs (b/L) curves for small (b/L) are all equal to minus one, with the ordinate distance between successive curves for various (a/h) ratios being inversely proportional to $(a/h)^{3/2}$

Although a point-for-point comparison is not presented herein, it is remarked that the analytical results given by Eq (12) for cylinders of moderate length are identical to those graphically illustrated by Almroth[2] in Figs 7a–7c for all values of (b/L) For purposes of further simplification and comparison, Eq (12) is now rewritten in terms of the well-known parameters of Batdorf [3]

Introducing the parameters k_y and Z_y, defined by

$$k_y = \bar{p}\;a12(1-\nu^2)/\lambda^2 Eh^3 \qquad Z_y = (L^2/ah)(1-\nu^2)^{1/2} \tag{13}$$

Eq (12) becomes

$$k_y = \frac{4(6)^{1/2}Z^{1/2}}{3[\pi b/L + \sin(\pi b/L)]} \tag{14}$$

This result is very interesting in that it reduces to the classical form of buckling predictions of moderately long cylinders un der a uniform external pressure Taking (b/L) equal to unity,

$$k_y = 1\;0395\;Z^{1/2} \tag{15}$$

The comments thus far have been confined to the moderately long cylinder range In Fig 1, a graphical representation of results for the short-cylinder range is presented in terms of Batdorf's parameters and various (b/L) ratios The graphical results are based on the analytical predictions of Eq (4) and are extended for completeness to include the moderately long cylinder range

From Fig 1, it is readily observed that the curves for various (b/L) ratios are parallel to the well-known curve for full uniform pressure loading $(b/L = 1)$ This observation leads to the important conclusion that the predictions for the critical band pressures are related to those for the uniform pres sure loading case by a simple scale factor A closer inspection of Eq (4), in addition to a re-examination of the definition for k_y, reveals the relationship

$$k_{yb} = \frac{k_{y0}}{\{(b/L) + [\sin(\pi b/L)/\pi]\}} \tag{16}$$

where the subscripts b and 0 have been attached to denote the band and uniform pressure loading cases, respectively

Thus, the final results for the predictions of critical band pressures have been reduced to a form that contains two nondimensional parameters; one parameter, k_{y0}, reflects the criti cal loading for full uniform pressure, and the other parameter is dependent on the ratio of band width to cylinder length

In the limiting case of a moderately long cylinder, the present results have been shown to be in agreement with the investigation by Almroth and Brush,[2] in which a reasonable

† It is briefly mentioned that the current predictions are based on the assumption of a large number of circumferential waves, i e , $n \gg 2$ Consequently, the present results are not expected to give an accurate measure of the critical pressures for problems that exhibit an oval-type instability

‡ Roughly speaking, this range is generally considered to be the moderate-length shell range and represents the parametric range for which results are given in Figs 7a–7c of Ref 2

correlation is obtained between theory and experimental data More experimental studies are necessary to confirm the validity of the predictions in the short cylinder range, even though the present results have been shown in this range to reduce to the classical predictions when (b/L) approaches unity

Befoie concluding, it is worthwhile to reiterate that the present investigation was restricted to a circumferential band loading that was uniformly distributed and equally spaced $(L/2)$ along the generator of the cylinder The analysis presented in Ref 1 is, however, sufficiently general so as to permit an extension of the theoretical results to circumferential band loadings, which vary along the generator of a simply supported cone or cylinder with arbitrary spacings The extension is straightforwaid and only involves a simple integration procedure once the appropriate step function has been determined to describe the loading condition under consideration

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Laminar Compressible Mixing behind Finite Bases

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IN Ref 1 Chapman considered the problem of laminar mixing at constant pressure for a fluid with Prandtl number one, and a viscosity power law of the form $\mu \sim T^{\omega}$ Because of the recent interest in the fluid-mechanic description of hypersonic wakes, mixing problems are being again in vestigated intensely both experimentally and theoretically As an example, Ref 2 extends Ref 1 by assuming a Blasius starting velocity profile rather than a uniform one (A uniform profile was assumed by Chapman as a necessity imposed for the conservation of similarity)

It will be shown in this brief note that the velocity along the dividing streamline can be determined in a simple manner by approximating the integral solution of Ref 1 The effect of finite base radius (or Reynolds number) is also investigated using this approximate solution

In terms of the stream function ψ, the differential equation of motion is

$$\frac{\zeta}{2}\frac{du}{d\zeta} + \frac{d}{d\zeta}\left(g\,\frac{du}{d\zeta}\right) = 0 \qquad (1)$$

where

$$\zeta = \psi/(U\nu sC)^{1/2} \qquad g(\zeta) = uT^{\omega-1}$$
$$T = T_d - [(\gamma - 1)/2]\, M^2u^2 + (T_0 - T_d)u$$

T and u are nondimensional with respect to the freestream, the subscripts d and 0 refer to the "dead water" region and stagnation point, and M is the freestream Mach number Chapman's boundary conditions are as follows:

$$\text{at } \zeta = \infty:\ u = 1 \qquad \zeta = -\infty:\ u = 0 \qquad (2)$$

The velocity at the dividing stieamline u_D is given after a formal solution of Eq (1) as follows:

$$u_D = \int_{\infty}^{0} F d\zeta \Big/ \left(\int_{-\infty}^{0} F d\zeta + \int_{0}^{+\infty} F d\zeta\right) \qquad (3)$$

where

$$F(\zeta) = \exp\left\{-\int_0^{\zeta} \frac{\zeta}{2g}\, d\zeta\right\}\Big/ g \qquad (4)$$

We make the observation that, in the expression for the integrand F, the contribution of the exponential part in the numerator is stronger than the denominator for increasing g (or ζ) Hence, in the interval I, $-\infty < \zeta \leq 0$, and in the interval II, $0 \leq \zeta < +\infty$, most of the contribution comes about from the values of g corresponding to the highest ζ inside the interval For a first iteration let us, therefore, assume for u the following step function: inside I, $u = u_D$, inside II, $u = 1$ Simple integration of Eq (3) yields

$$u_D = \frac{1}{1 + (u_D T_D^{\omega-1})^{1/2}} \qquad (5)$$

Assuming that $T_d = T_0$ and $\omega = 0\ 75$ the calculations show that, for $M = 0, 1, 2, 3, 4, 5, 7, 10, 15, 20$, the corresponding values of u_D are 0 570, 0 573, 0 581, 0 590, 0 598, 0 605, 0 619, 0 634, 0 652, 0 664 For $M = 0$ and 5, Chapman[3] gives, through an exact numerical solution, $u_D = 0\ 587$ and 0 597 Comparison shows that our closed-form approximation is in error of less than -3% and $+1\ 5\%$, correspondingly

Experiments show[4] that u_D is a function of Reynolds number The analysis of Ref 2, in which the influence of an initial finite boundary-layer thickness was studied, through the parameter $s^* \sim s/s_b$ yields results that are independent of the base radius r_0 (This occurs because, as can be seen from Fig 1, $r_0 = s_b \sin\alpha = s_n \sin\beta$) One might conjecture that one way to introduce the Reynolds number would be to assume that $u = 0$ not at $\zeta = -\infty$ but at $\zeta = -\zeta_n$, where ζ_n is finite and positive This assumption implies a finite radius in the direction perpendicular to the main flow Following the same method used for the derivation of Eq (5), we find

$$u_D = \text{erf}(\chi_n)/[\text{erf}(\chi_n) + (g_D)^{1/2}] \qquad (6)$$

where $\chi_n = \zeta_n/2\ (g_D)^{1/2}$, and erf denotes the error function Figure 1 shows the function $u_D(M,\zeta_n)$ for three different Mach numbers

These results are best interpreted in the physical plane s,y For simplicity assume $M = 0$, so that[1]

$$y\left(\frac{u}{\nu s}\right)^{1/2} = \int_0^{\zeta} \frac{d\zeta}{u} \qquad (7)$$

Let ζ_n correspond to y_n and s_n, where the subscript n indicates the position of the "neck" of thickness h as shown† in Fig 2 To calculate the integral in Eq (7), we approximate $u(\zeta)$ in the interval $0 \leq \zeta \leq \zeta_n$ by dropping the first term in Eq (1) A simple integration yields

$$u = u_D\ [1 - (\zeta/\zeta_n)]^{1/2} \qquad (8)$$

Introducing the foregoing into Eq (7), we have

$$y_n(u/\nu s_n)^{1/2} = 2(\zeta_n/u_D) \qquad (9)$$

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† A correction of this geometry to take into account the weaker growth of the boundary layer above the dividing streamline is very small